Codes on Graphs and the Pseudocodewords

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Notation

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Given a parity check matrix H of C and $\mathbf{y} \in \mathbb{F}_2^n$,

$$\mathbf{y} \in C \text{ if and only if } H\mathbf{y}^T = \mathbf{0}.$$

Parity check matrix of a code is not unique.

The code

$$C = \left\{ \begin{array}{ll} (0,0,0,0), & (0,1,1,1) \\ (1,0,1,0), & (1,1,0,1) \end{array} \right\} \subset \mathbb{F}_2^4$$

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$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

A code C given by a parity check matrix H is denoted $C(H)$.

A parity check matrix corresponds to a bipartite graph called the Tanner graph.

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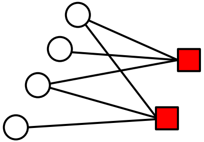

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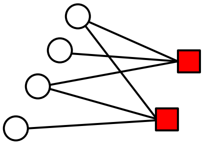

The **Tanner graph** of H is a bipartite graph with a vertex set $X \cup F$ such that:

- Each vertex in X corresponds to a column of H and is called a bit node.
- Each vertex in F corresponds to a row of H and is called a check node.
- $\{x_i, f_j\}$ is an edge if and only if $h_{ji} = 1$.

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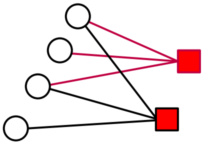

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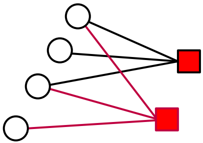

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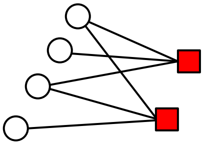

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A codeword corresponds to a valid configuration on the Tanner graph.

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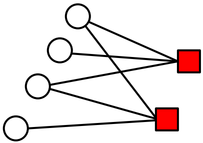

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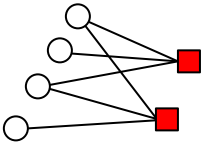

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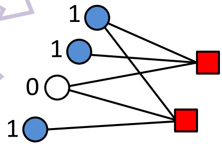

Since a check node represents a row, i.e. a parity condition, of H, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is a codeword of $C(H)$ if and only if every check node is adjacent to an even number of 1's.

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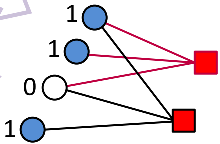

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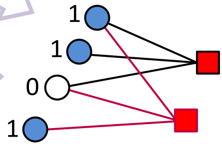

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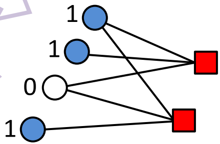

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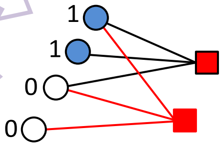

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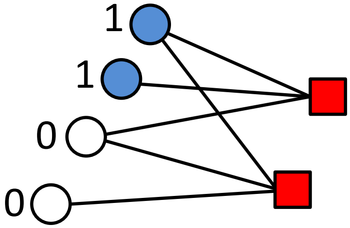
Thou shall vote!



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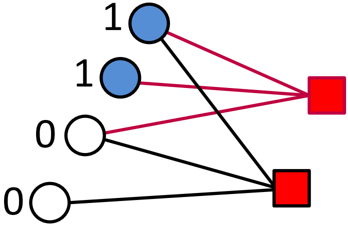

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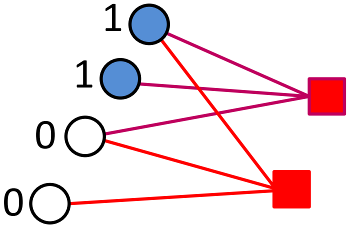

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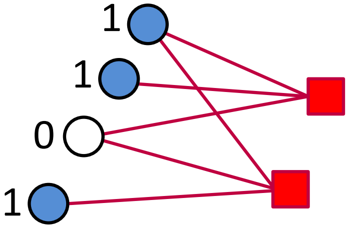

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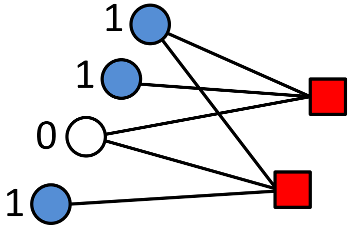

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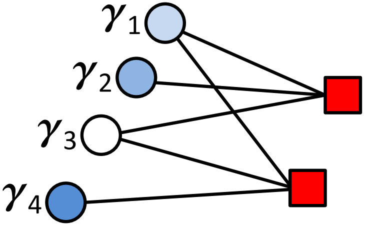
Gallager's algorithm A reassigns the value of the bit notes that are adjacent to a certain number of unsatisfied check nodes.

Soft decision decoding works well on graphs.



Upon receiving the message $\mathbf{w}$, initialize each bit with $\gamma_i = \log\left(\frac{P(w_i|c_i=0)}{P(w_i|c_i=1)}\right)$, the log-likelihood ratio at the i^{th} coordinate.

The maximum likelihood decoding is equivalent to finding a binary value assignment $(c_1, c_2, \ldots, c_n)$ to the bit nodes such that $\sum_{i=1}^{n} \gamma_i c_i$ is minimized and every check node is adjacent to an even number of 1's.

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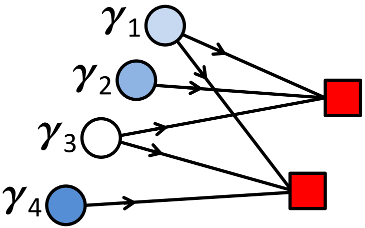

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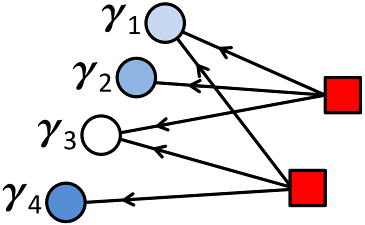

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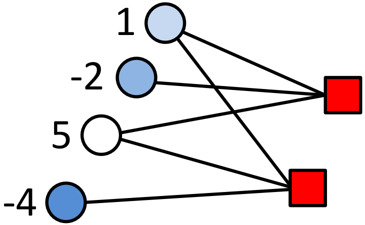

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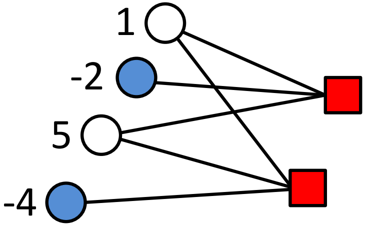

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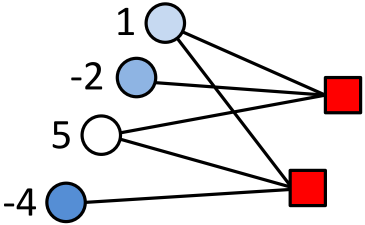

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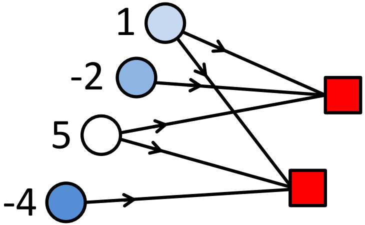

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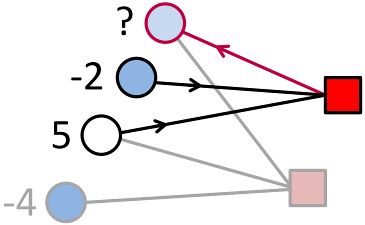

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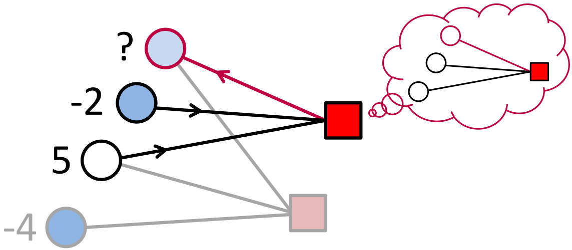

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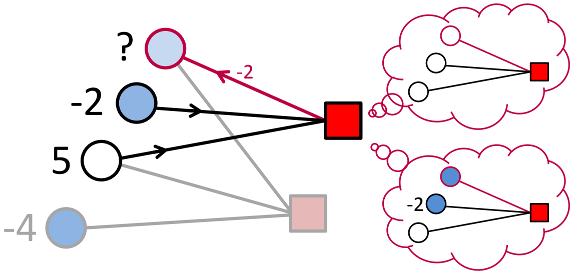

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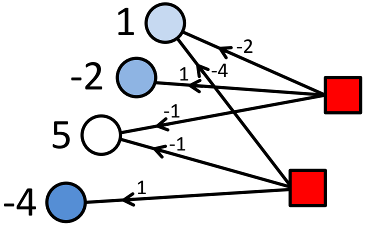

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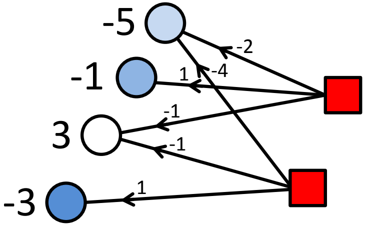

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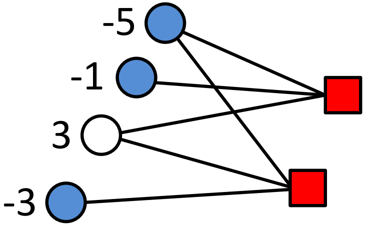

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Min-sum algorithm marginalizes the cost function.

- Initialization: For each bit node i, initialize the local cost γ_i. For each check node j and for all $s \in Nbhd(j)$, initialize $\mu_{j,s}^{(0)} := 0$.
- Iteration: For $i = 1, \ldots, m$:
  1. For all bit nodes s and for all $j \in Nbhd(s)$, bit-to-check messages are given by
  $$\mu_{s,j}^{(i)} := \gamma_s + \sum_{j' \in Nbhd(s)-\{j\}} \mu_{j',s}^{(i-1)}.$$
  2. For all check nodes j and for all $s \in Nbhd(j)$, check-to-bit messages are given by
  $$\mu_{j,s}^{(i)} := \prod_{s' \in Nbhd(j)-\{s\}} \operatorname{sgn}\left(\mu_{s',j}^{(i)}\right) \cdot \min_{s' \in Nbhd(j)-\{s\}} \left|\mu_{s',j}^{(i)}\right|$$
- Final cost computation: The final cost at the bit node i after m iterations is
$$\mu_i := \gamma_i + \sum_{j \in Nbhd(i)} \mu_{j,i}^{(m)}.$$

Sum-product algorithm marginalizes the probability.

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  $$\mu_{j,s}^{(i)} := \log\left(\frac{1+\prod_{s' \in Nbhd(j)-\{s\}} \tanh\left(\mu_{s',j}^{(i)}/2\right)}{1-\prod_{s' \in Nbhd(j)-\{s\}} \tanh\left(\mu_{s',j}^{(i)}/2\right)}\right)$$
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Advantages and disadvantages of iterative decoders

- Easy to implement.
- Can be applied to any code.
- Low-complexity: exponential in the check node degree, linear in code length for low-density parity-check (LDPC) codes.
- Allow communication at rates near the channel capacity.

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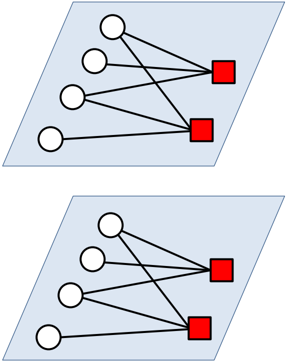

A **graph cover of degree** m of the Tanner graph of H is a bipartite graph $\widetilde{G}$ such that for each vertex v there is a set of vertices $\{v_1, \ldots, v_m\}$ of $\widetilde{G}$ with $\deg v_i = \deg v$ for all $1 \leq i \leq m$, and for every edge $\{u, v\} \in E$ there are m edges from $\{u_1, \ldots, u_m\}$ to $\{v_1, \ldots, v_m\}$ connected in a 1-1 manner.

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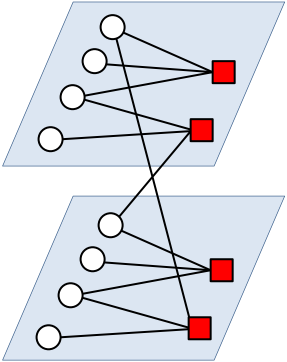

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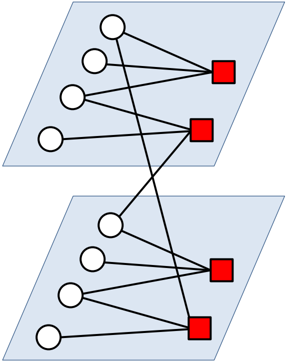

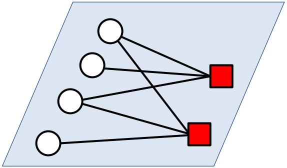

Iterative decoders operate locally on the Tanner graph, and so they cannot distinguish between the code defined by the graph cover of H and the code defined by H.

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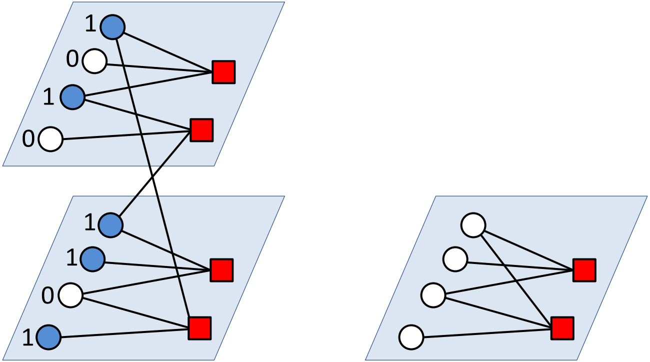

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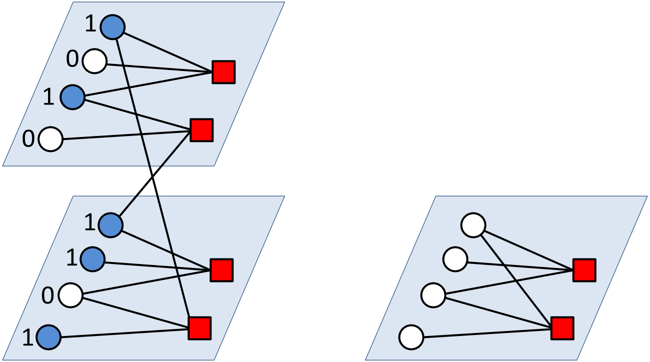

Iterative decoders operate locally on the Tanner graph, and so they cannot distinguish between the code defined by the graph cover of H and the code defined by H.

Let $C(\widetilde{G}) \subset \mathbb{F}_2^{mn}$ be the code defined by a graph cover $\widetilde{G}$ of the Tanner graph of H.

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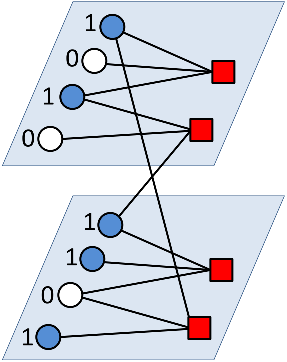

A **pseudocodeword** of $C(H)$ is $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ such that there exists a graph cover $\widetilde{G}$ of degree m and a codeword

$$\left(c_{(1,1)}, \ldots, c_{(1,m)}; \ldots; c_{(n,1)}, \ldots, c_{(n,m)}\right) \in \mathbb{F}_2^{mn}$$

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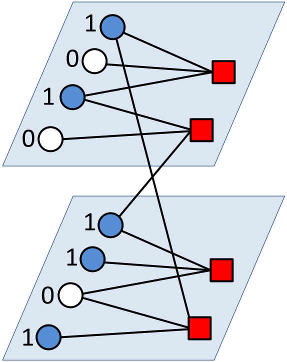

For example,

$$(2, 1, 1, 1)$$

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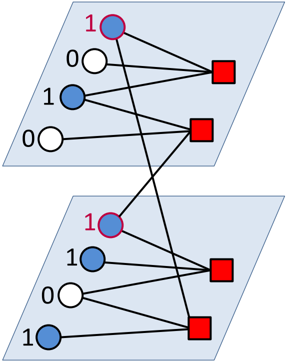

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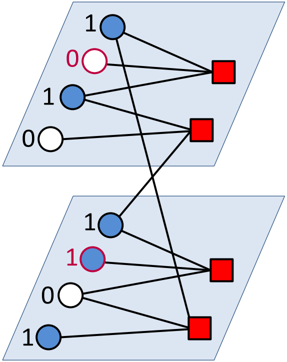

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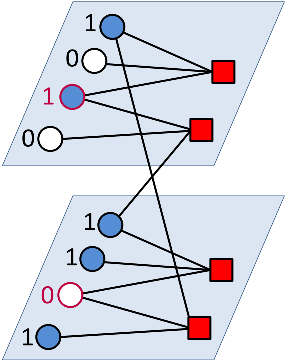

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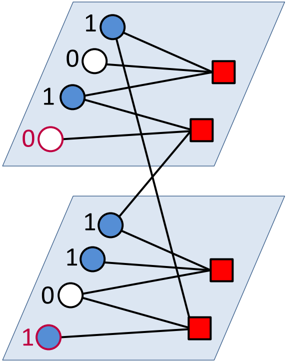

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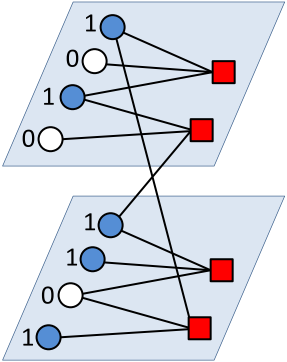

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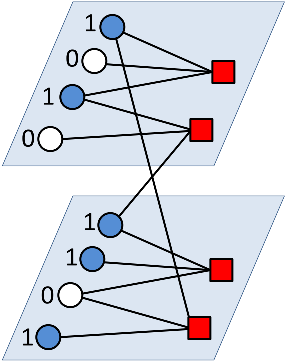

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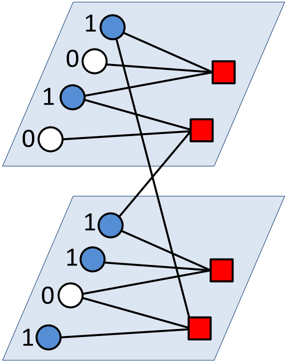

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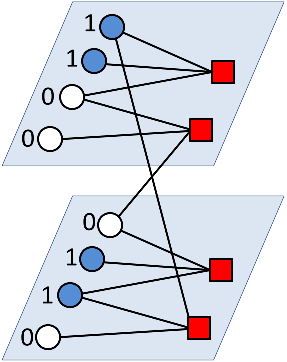

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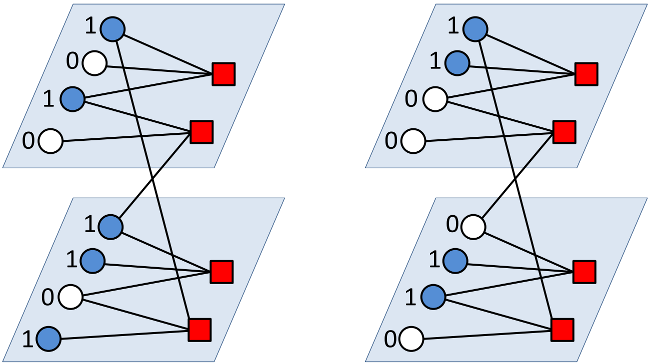

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Let

$$P(H) = \{\mathbf{p} \in \mathbb{N}^n \mid \mathbf{p} \text{ is a pseudocodeword of } C(H)\}.$$

Given H, the set of pseudocodewords can be completely characterized via the fundamental cone.

Definition

The **fundamental cone** of a parity check matrix $H \in \mathbb{F}_2^{r \times n}$ is

$$\mathcal{K}(H) = \left\{ \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i \geq 0 \text{ and } Row_j(H)\mathbf{v}^T \geq 2h_{ji}v_i \; \forall i, j \right\}.$$

Theorem (Koetter, Li, Vontobel, and Walker 2007)

Given $H \in \mathbb{F}_2^{r \times n}$, $\mathbf{p} \in \mathbb{N}^n$ is a pseudocodeword of $C(H)$ if and only if

- $\mathbf{p} \in \mathcal{K}(H)$, and
- $\mathbf{p}$ reduces mod 2 to a codeword.

The generating function for the pseudocodewords of a cycle code is an edge zeta function.

Consider the pseudocodeword enumerator

$$\sum_{\mathbf{v} \in P(H)} \mathbf{x}^{\mathbf{v}}$$

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Theorem (Koetter et al. 2007)

If $C(H)$ is a cycle code, a code for which the parity check matrix H has exactly two 1's in each column, then the following are equivalent:

- $(p_1, p_2, \dots, p_n)$ is a pseudocodeword of $C(H)$.
- $x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$ has a nonzero coefficient in the power series expansion of the edge zeta function $\zeta(x_1, x_2, \dots, x_n)$ of the normal graph of H, which is a rational function.

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Question: Can we generalize this?

The set of pseudocodewords itself does not form a cone.

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Given a parity check matrix $H \in \mathbb{F}_2^{r \times n}$, the **lifted fundamental cone** of $C(H)$ is

$$\hat{\mathcal{K}}(H) = \left\{ (\mathbf{v}, \mathbf{a}) \in \mathbb{R}^{n+r} \middle| \begin{array}{l} H\mathbf{v}^T = 2\mathbf{a}^T, \\ v_i \geq 0, \text{and } Row_j(H)\mathbf{v}^T \geq 2h_{ji}v_i \ \forall i, j \end{array} \right\}.$$

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Proposition

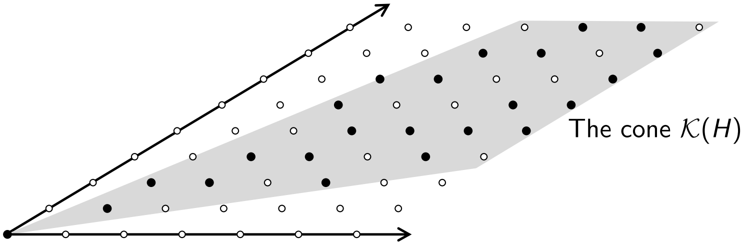
Let $H \in \mathbb{F}_2^{r\times n}$. Then

$$\pi\left(\hat{\mathcal{K}}(H)\right) = \mathcal{K}(H),$$

and the set of pseudocodewords of $C(H)$ is

$$\pi\left(\hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r}\right) = P(H).$$

The lifted fundamental cone sieves out precisely the the lattice points in $\mathcal{K}(H)$ which are not pseudocodewords.



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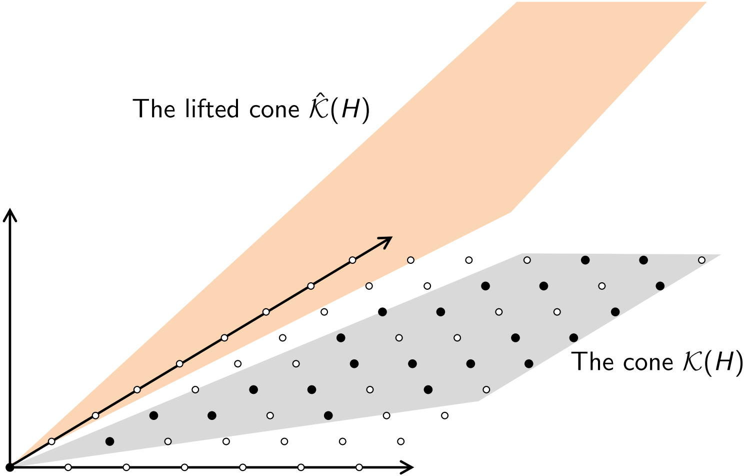

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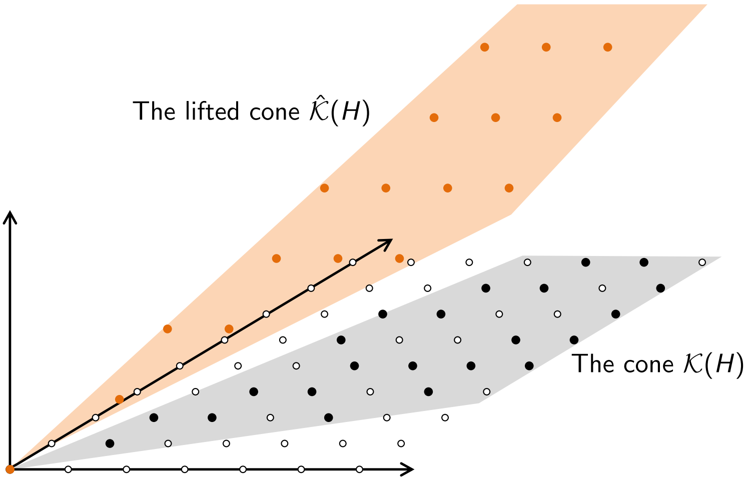

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Denote the generating function for integer points in the lifted fundamental cone

$$f(x_1, x_2, \ldots, x_{n+r}) := \sum_{(\mathbf{v},\mathbf{a}) \in \hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r}} \mathbf{x}^{(\mathbf{v},\mathbf{a})}.$$

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Specializing the above function at $(x_1, \ldots, x_n, 1, \ldots, 1)$ yields

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Theorem (K. and Matthews 2011)

Given $H \in \mathbb{F}_2^{r \times n}$, the generating function for the pseudocodewords of $C(H)$, $\sum_{\mathbf{v} \in P(H)} \mathbf{x}^{\mathbf{v}}$, is a rational function.

Barvinok's algorithm produces the generating function of a cone as a rational function.

Theorem (Barvinok 1994)

Fix the dimension d. Given a rational cone $K \subset \mathbb{R}^d$, there exists a polynomial time algorithm which computes the generating function $\sum_{\mathbf{a} \in K \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{a}}$ of the form

$$\sum_{i \in I} \epsilon_i \frac{\mathbf{x}^{\mathbf{u}_i}}{(1 - \mathbf{x}^{\mathbf{u}_{i1}}) \cdots (1 - \mathbf{x}^{\mathbf{u}_{id}})}$$

where $\epsilon_i \in \{1, -1\}$, and $\mathbf{u}_i, \mathbf{u}_{ij}$ are integer vectors.

Example

Consider the code $C(H)$ where $H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

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$$\sum_{\mathbf{p} \in P(H)} \mathbf{x}^{\mathbf{p}} = \frac{1 - x_1^2 x_2^2 x_3^2 x_4^2}{(1 - x_1 x_3 x_4^2)(1 - x_1 x_2^2 x_3)(1 - x_2 x_3 x_4)(1 - x_1 x_2 x_4)(1 - x_1 x_3)}$$

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$$\sum_{\mathbf{p} \in P(H)} \mathbf{x}^{\mathbf{p}} = \frac{1 - x_1^2 x_2^2 x_3^2 x_4^2}{(1 - x_1 x_3 x_4^2)(1 - x_1 x_2^2 x_3)(1 - x_2 x_3 x_4)(1 - x_1 x_2 x_4)(1 - x_1 x_3)}$$

$$= 1 + x_1 x_3 + x_2 x_3 x_4 + x_1 x_2 x_4 + x_1 x_3 x_4^2 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1^2 x_2 x_3 x_4 + x_1 x_2 x_3^2 x_4 + \dots$$

The pseudocodewords of $C(H)$ are

$$(0,0,0,0), (1,0,1,0), (0,1,1,1), (1,1,0,1), (1,0,1,2), (1,2,1,0),$$

$$(2,0,2,0), (2,1,1,1), (1,1,2,1), \dots$$

Example

Consider the code $C(H)$ of length 7 and dimension 2 given by a parity check matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

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$$\sum_{\mathbf{p} \in P(H)} \mathbf{x}^{\mathbf{p}} = \frac{1}{(1 - x_1 x_2 x_3)(1 - x_1 x_2 x_3 x_4^2 x_5 x_6 x_7)(1 - x_5 x_6 x_7)}.$$

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This gives a complete characterization of the pseudocodewords of $C(H)$; that is, $P(H)$ can be written as

$$\{a(1,1,1,0,0,0,0) + b(1,1,1,2,1,1,1) + c(0,0,0,0,1,1,1) \mid a, b, c \in \mathbb{Z}\}.$$

Irreducible pseudocodewords are those most likely to cause decoding error.

A nonzero pseudocodeword is said to be **irreducible** provided it cannot be written as a sum of two or more nonzero pseudocodewords.

Given a parity check matrix $H \in \mathbb{F}_2^{r \times n}$, let $\mathrm{Irr}(H)$ denote the set of irreducible pseudocodewords of $C(H)$.

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The **Hilbert basis** of an additive semigroup $(G,+)$ is the minimal set of elements $\{b_1,\ldots,b_t\}$ such that

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Theorem (K. and Matthews 2011)

Given $H \in \mathbb{F}_2^{r \times n}$, the set of integer points in the lifted fundamental cone $\hat{\mathcal{K}}(H)$ forms a semigroup under addition.

Furthermore, if $\mathfrak{B}$ is the Hilbert basis of $\hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r}$, then

$$\mathrm{Irr}(H) = \pi(\mathfrak{B}).$$

Example

Consider the simplex code of length 7 and dimension 3 with two choices for parity check matrix

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \color{red}{1} & \color{red}{0} & \color{red}{0} & 1 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \color{red}{0} & \color{red}{1} & \color{red}{1} & 1 \end{pmatrix}.$$

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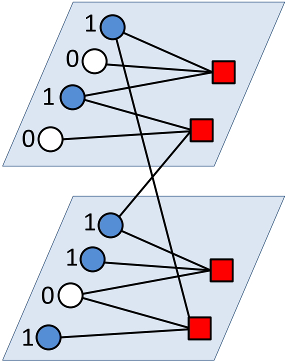
The noncodeword irreducible pseudocodewords, found using 4ti2, are

$$\underbrace{\underbrace{\begin{array}{ll}
(0,0,0,2,2,2,2), & (0,3,0,1,2,1,1), \\
(0,0,3,2,1,1,1), & (0,0,1,2,1,1,1), \\
(3,0,0,1,1,2,1), & (0,1,0,1,2,1,1), \\
(2,1,0,1,0,1,1), & (1,2,0,1,1,0,1), \\
(2,0,1,0,1,1,1), & (0,2,1,0,1,1,1), \\
(1,0,0,1,1,2,1), & (0,1,2,1,0,1,1), \\
(1,0,2,1,1,0,1), &
\end{array}}_{\text{Irreducible pseudocodeword of } C(H_1)}
\begin{array}{lll}
(2,0,3,0,1,1,1), & (4,0,0,0,2,2,2), & (0,0,0,4,2,2,2), \\
(0,4,0,0,2,2,2), & (2,0,0,0,2,2,2), & (0,2,0,0,2,2,2), \\
(3,0,0,3,1,2,1), & (1,1,0,2,1,1,0), & (3,3,0,0,1,1,2), \\
(1,1,0,0,1,1,2), & (0,1,0,3,2,1,1), & (1,3,0,0,1,1,2), \\
(3,1,0,0,1,1,2), & (1,0,0,3,1,2,1), & (0,0,0,0,2,2,2), \\
(0,3,0,3,2,1,1), & (1,1,2,0,1,1,0), & (0,0,3,0,1,1,1), \\
(0,2,3,0,1,1,1). & &
\end{array}}_{\text{Irreducible pseudocodeword of } C(H_2)}$$

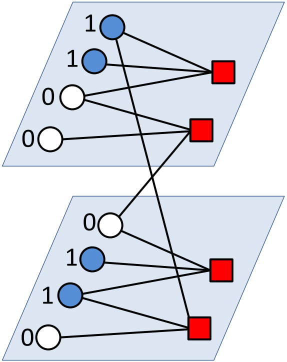
Question: Given a code C, what is a good choice of parity check matrix for C?

Some pseudocodewords are sums of codewords.



$(2, 1, 1, 1)$
$= (1, 0, 1, 0) + (1, 1, 0, 1)$

Some are not.



$(1, 2, 1, 0)$
$\neq$ any sum of codewords

A geometrically perfect code has no "bad" pseudocodewords.

Given $H \in \mathbb{F}_2^{r \times n}$,

$$\left\{ \sum_{\mathbf{c} \in C(H)} a_{\mathbf{c}} \mathbf{c} \;\middle|\; a_{\mathbf{c}} \in \mathbb{N} \right\} \subseteq P(H),$$

where $\sum_{\mathbf{c} \in C(H)} a_{\mathbf{c}} \mathbf{c} \in \mathbb{N}^n$.

If the equality holds, $C(H)$ is called **geometrically perfect**.

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Lemma (Wiberg 1996)

If H is cycle-free, then $C(H)$ is geometrically perfect.

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If H is cycle-free, then $C(H)$ is geometrically perfect.

Sketch of Proof: A graph cover of H is disconnected copies of H.

Theorem (K. 2012)

If C is a code which can be represented by a cycle-free parity check matrix, then the following are equivalent:

1. $C(H)$ is geometrically perfect.
2. There exist rows $s_1, s_2, \ldots, s_t$ of H such that

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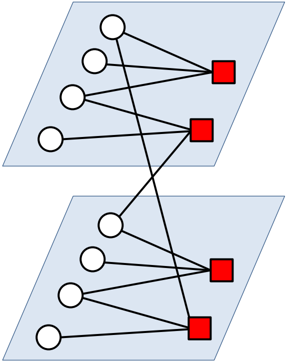
   is cycle-free and $C(T) = C(H)$.

To study pseudocodewords, we define pseudocheck.



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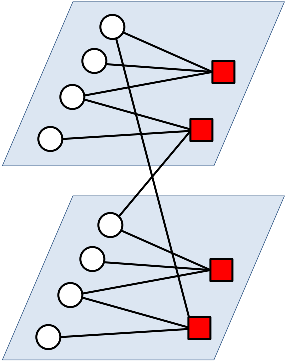

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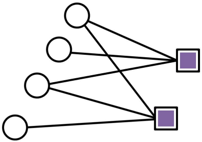

A **pseudocheck** node is satisfied if and only if the integer value assignment $(a_1, \ldots, a_t)$ to its neighbors satisfies the following conditions:

- $a_i \geq 0$ for all i,
- $\sum_{j=1}^{t} a_j = 0 \mod 2$, and
- $\sum_{j=1, j\neq i}^{t} a_j \geq a_i$ for all i.

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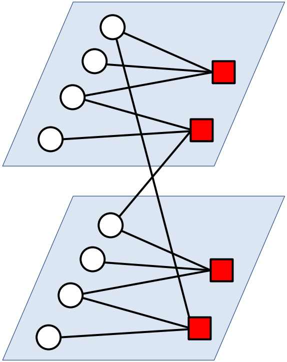

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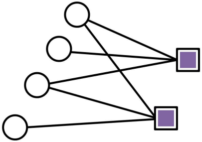

An integer vector **p** is a pseudocodeword if and only if every pseudocheck is satisfied.

Theorem

If C is a code which can be represented by a cycle-free parity check matrix, then the following are equivalent:

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If $T = H - \{s_1, s_2, \ldots, s_t\}$, then $P(H) \subseteq P(T)$.

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Lemma (Kelley and Sridhara 2007)

If $T = H - \{s_1, s_2, \ldots, s_t\}$, then $P(H) \subseteq P(T)$.

Proof (2. $\Rightarrow$ 1.): If T is cycle free and $C(T) = C(H)$, then

$$P(H) \subseteq P(T) = \left\{ \sum_{\mathbf{c} \in C(T)} a_{\mathbf{c}} \mathbf{c} \;\middle|\; a_{\mathbf{c}} \in \mathbb{N} \right\} = \left\{ \sum_{\mathbf{c} \in C(H)} a_{\mathbf{c}} \mathbf{c} \;\middle|\; a_{\mathbf{c}} \in \mathbb{N} \right\} \subseteq P(H).$$

Therefore, $C(H)$ is geometrically perfect.

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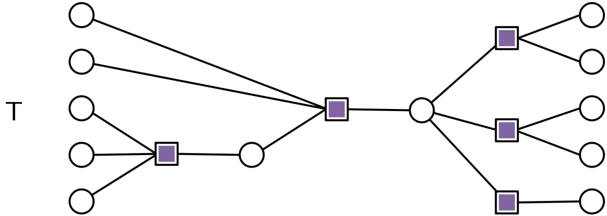
The code C can be represented by a cycle-free parity check matrix T.



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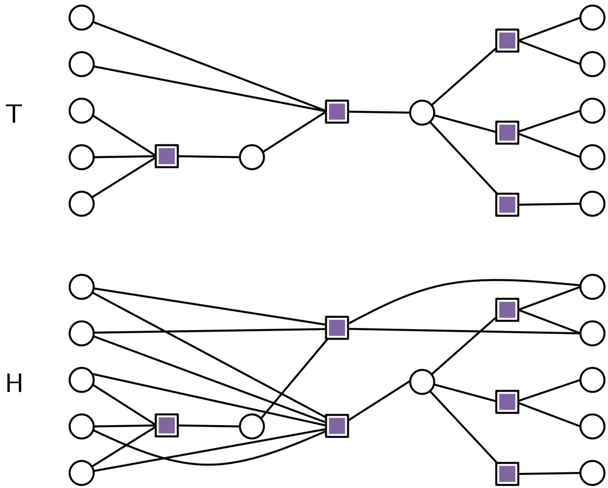

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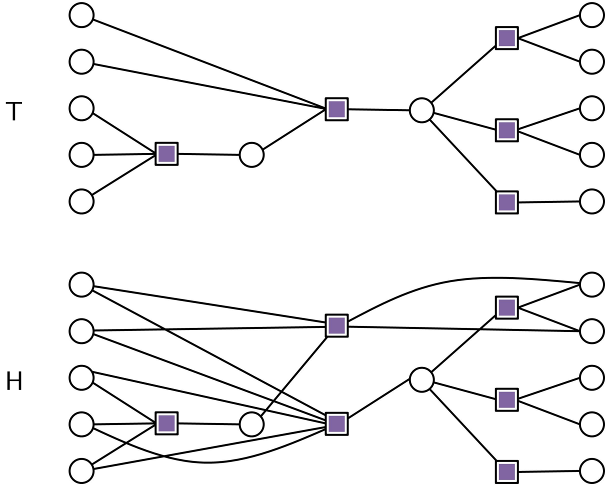
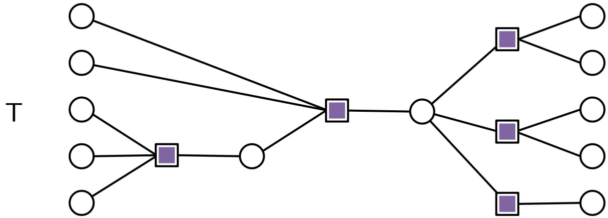


There exists a pseudocheck in T which is not in H.

A pivotal pseudocheck in T cannot be replaced by a pseudocheck from H.



A pseudocheck u of T is **pivotal** if there does not exist a pseudocheck h of H such that $(T - \{u\}) \cup \{h\}$ is cycle-free and $C((T - \{u\}) \cup \{h\}) = C(T)$.

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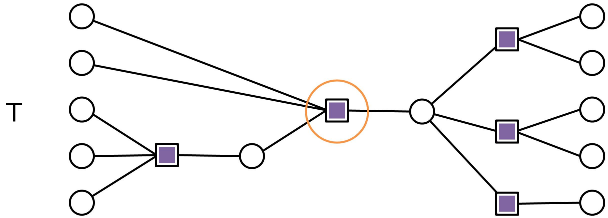

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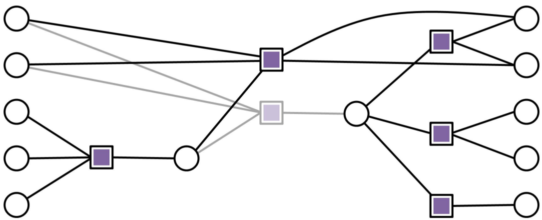

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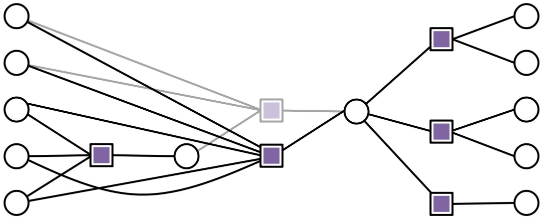

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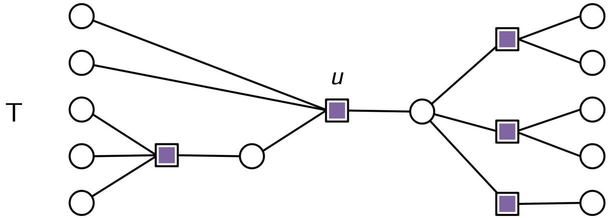
Construct a pseudocodeword from a pivotal pseudocheck.



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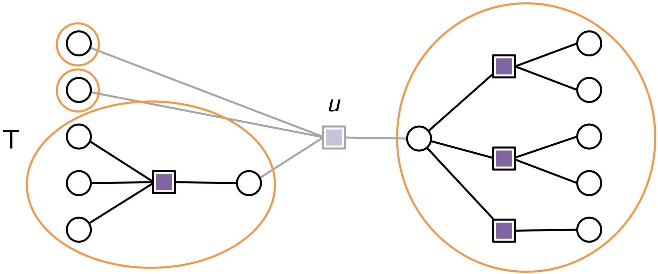

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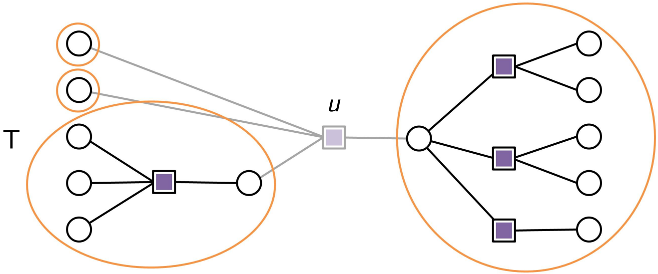


Fix a pivotal pseudocheck u.
The graph of $T - \{u\}$ has several connected components.
Assign the value $2\deg(u)$ to the bit nodes in one component, 0 to the bit nodes adjacent to a pseudocheck of degree 1, and 2 to all other bit nodes.

Claim

The assignment is valid for H, but not for T.

Proof of Claim: The assignment is invalid for T because it violates pseudocheck u.

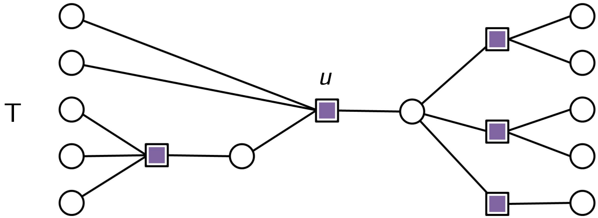
Fact

Suppose that $\tau_1, \ldots, \tau_{\deg(u)}$ are connected components in $T - \{u\}$ where u is adjacent to a bit node in $\tau_1, \ldots, \tau_{\deg(u)}$ in T.
For any pseudocheck h in H, if $\deg\left(h\mid_{\tau_i}\right) \geq 1$ for some i, then either

- $\deg\left(h\mid_{\tau_i}\right) \geq 2$, or
- $\deg(h) \geq \deg(u) + 1$.

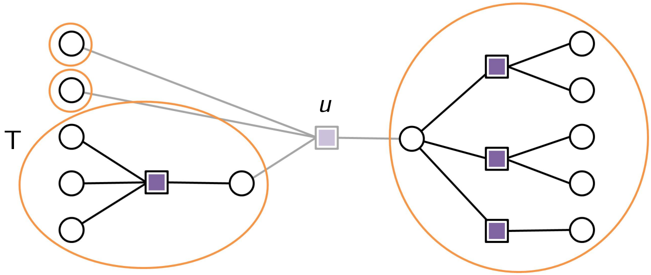
So, the assignment is valid for H.

Example



The pivotal pseudocheck u has degree 4.

Example



The pivotal pseudocheck u has degree 4.
There are 4 connected components in $T - \{u\}$.

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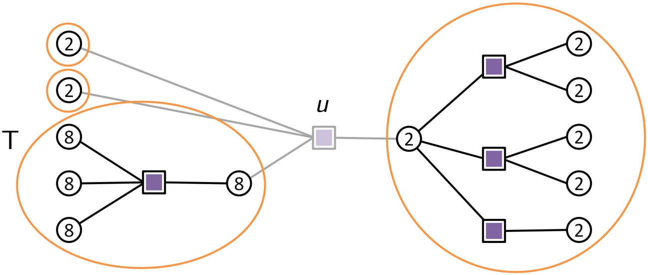

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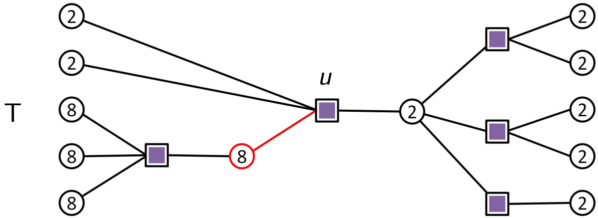
Example



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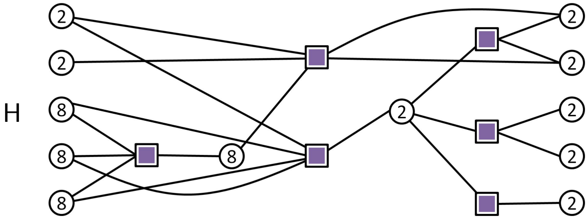

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Theorem (K. 2012)

If C is a code which can be represented by a cycle-free parity check matrix, then the following are equivalent:

1. $C(H)$ is geometrically perfect.
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- There exists a class of parity check matrices with many small cycles that perform well under iterative decoders.

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- There exists a class of parity check matrices with many small cycles that perform well under iterative decoders.
- A code C is capable of being geometrically perfect if and only if $\mathcal{H}_7^{\perp}$, $\mathcal{R}_{10}$, or $C(\mathcal{K}_5)^{\perp}$ cannot be obtained from C via a sequence of shortening and puncturing operations (Kashyap 2008).

Table of contents

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$\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the finite field with p elements where p is prime. $\oplus$ and $\odot$ denote finite field addition and multiplication.

A **linear code** C of length n and dimension k over $\mathbb{F}_p$ is a subspace of $\mathbb{F}_p^n$ of dimension k.

A **parity check matrix** of a code C is any matrix $H \in \mathbb{F}_p^{r \times n}$ such that C is the null space of H.

Given a parity check matrix H of C and $\mathbf{y} \in \mathbb{F}_p^n$,

$$\mathbf{y} \in C \text{ if and only if } H \odot \mathbf{y}^T = \mathbf{0} \in \mathbb{F}_p^{r \times 1}.$$

Denote $C(H)$ the code given by a parity-check matrix H.

The Tanner graph of a nonbinary parity-check matrix is a graph with weighted edges.

$$H = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{pmatrix} \in \mathbb{F}_3^{2\times 4}$$

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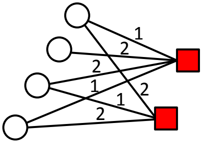

The bit nodes $X = \{x_1, \ldots, x_n\}$ correspond to a column of H, the check nodes $F = \{f_1, \ldots, f_r\}$ correspond to a row of H, and if $h_{ji} \neq 0$ then $\{x_i, f_j\}$ is an edge with weight

$$w(x_i, f_j) = h_{ji}.$$

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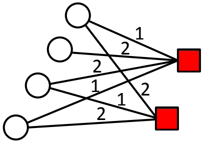

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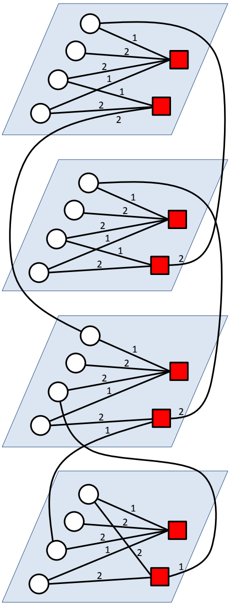
A vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{F}_p^n$ is a codeword of $C(H)$ if and only if

$$\sum_{i \in Nbhd(f_j)} w(x_i, f_j) \odot c_i = 0$$

for all j where the summation is taken over $\mathbb{F}_p$.

The Tanner graph of a nonbinary code permits graph covers similar to the Tanner graph of a binary code.



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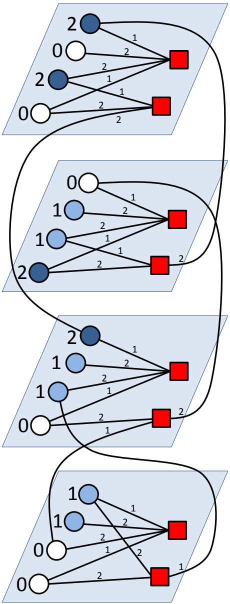

A pseudocodeword of $C(H)$ is a matrix $M \in \mathbb{Z}^{p-1 \times n}$ such that there exists a graph cover $\widetilde{G}$ and a codeword $\tilde{\mathbf{c}}$ of $C(\widetilde{G})$ where

$$m_{bi} := |\{1 \leq l \leq m \mid c_{(i,l)} = b\}|$$

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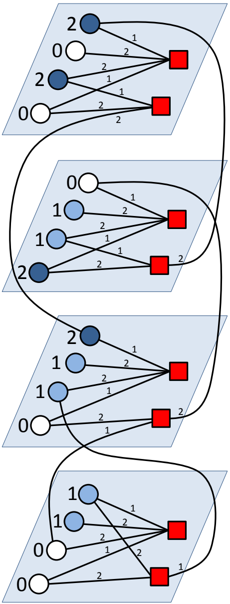

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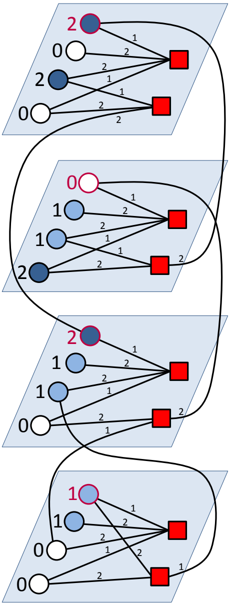

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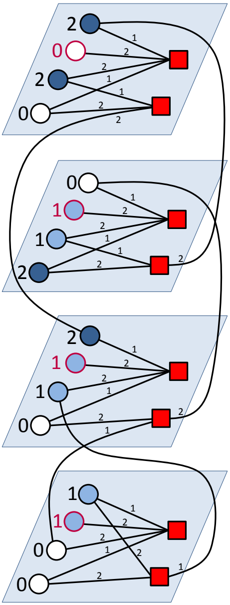

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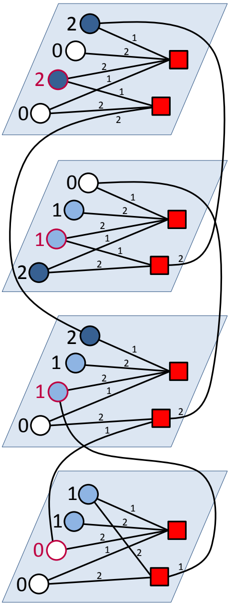

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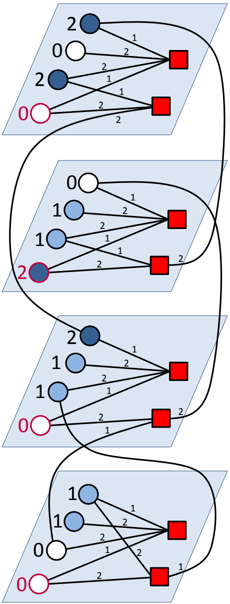

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We aim to generalize the binary fundamental cone.

Definition of Binary Fundamental Cone

The fundamental cone of a parity check matrix $H \in \mathbb{F}_2^{r\times n}$ is

$$\mathcal{K}(H) = \left\{ \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i \geq 0 \text{ and } Row_j(H)\mathbf{v}^T \geq 2h_{ji}v_i \;\forall i, j \right\}.$$

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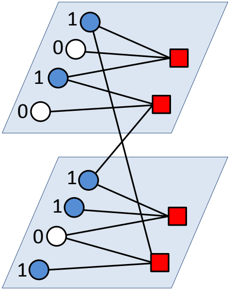

If $\mathbf{v}$ is a pseudocodeword,

$$Row_j(H)\mathbf{v}^T = \sum_{i=1}^{n} h_{ji} v_i = \sum_{i=1}^{n} \sum_{l=1}^{m} h_{ji} c_{(i,l)}.$$

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$$\begin{aligned}
\Theta_j &= \sum_{i=1}^{n} \sum_{l=1}^{m} h_{ji} \odot c_{(i,l)} \\
&= \sum_{i=1}^{n} \sum_{b=1}^{p-1} \sum_{\{l \mid c_{(i,l)}=b\}} b \odot h_{ji} \\
&= \sum_{b=1}^{p-1} \sum_{i=1}^{n} \left(\left(b \odot h_{ji} \right) \cdot \sum_{\{l \mid c_{(i,l)}=b\}} 1 \right) \\
&= \sum_{b=1}^{p-1} \sum_{i=1}^{n} \left(b \odot h_{ji} \right) m_i(b) \\
&= \sum_{b=1}^{p-1} \left(b \odot Row_j(H) \right) Row_b(M)^T.
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If $\mathbf{c}$ is a codeword of a p-ary code C, so is $a \odot \mathbf{c}$ where $a \in \mathbb{F}_p^*$.

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If **c** is a codeword of a p-ary code C, so is $a \odot \mathbf{c}$ where $a \in \mathbb{F}_p^*$.

Definition

Define

$$\Theta_j(a, M) = \sum_{b=1}^{p-1} \Big(a \odot b \odot Row_j(H) \Big) Row_b(M)^T$$

where $a \in \mathbb{F}_p^*$, $1 \leq j \leq r$, and $M \in \mathbb{Z}^{(p-1)\times n}$.

Theta function gives bound similar to the binary case.

Recall the inequality $\frac{1}{2} Row_j(H)\mathbf{v}^T \geq h_{ji} v_i$

Proposition

Consider a parity-check matrix $H \in \mathbb{F}_p^{r \times n}$ where p is prime. If M is a pseudocodeword of $C(H)$, then

$$\frac{1}{p}\Theta_j(a, M) \geq m(1) + m(2) + \cdots + m(p-1)$$

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This inequality, however, is insufficient.

Critical multiset is the "forbidden" configurations.

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$$\sum_{\gamma_i \in \Gamma} \gamma_i \neq 0 \mod p,$$

and any multisubset of Γ is critical.

Toward characterizing nonbinary pseudocodewords

Theorem (K. and Matthews 2012)

Let $H \in \mathbb{F}_p^{r\times n}$ where p is prime. If M is a pseudocodeword of $C(H)$, then

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for all $a, b \in \mathbb{F}_p^*$, $1 \leq j \leq r$, $i, i_1, \ldots, i_t \in supp(Row_j(H))$, and critical multiset $\{\gamma_1, \ldots, \gamma_t\}$.

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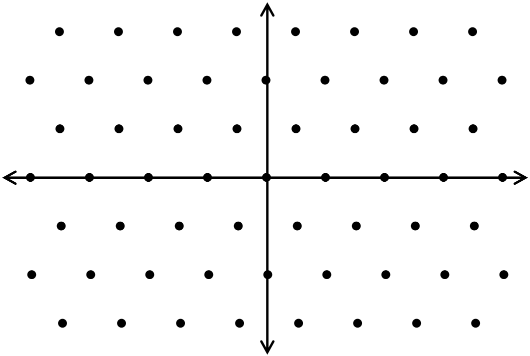
- The converse is true for binary codes (Koetter et al. 2007) and ternary codes (Skachek 2010).
- The number of inequalities is exponential in n.

Lattice codes are linear codes analog for continuous-valued AWGN channel.



A lattice is a discrete additive subgroup of $\mathbb{R}^n$.

We may apply iterative decoders to lattices constructed from codes using Construction A.

A binary code $C(H) \subseteq \mathbb{F}_2^n$ yields a lattice

$$\Lambda_A = \{\mathbf{v} \in \mathbb{Z}^n \mid \mathbf{v} \text{ reduces mod 2 to a codeword of } C(H)\}.$$

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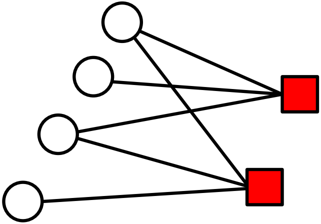

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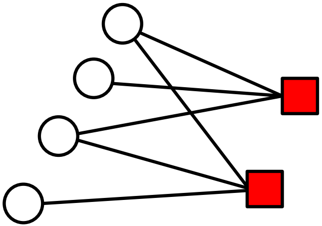
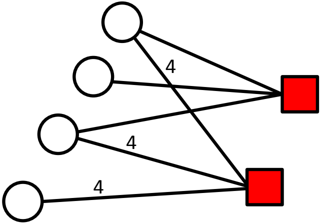


A check node is satisfied if the sum of its neighbors is an even integer. Apply iterative decoders for binary codes to $\mathbf{v} - \mathbf{a} \in [0,1]^n$ for an appropriately chosen $\mathbf{a}$ (Conway and Sloane 1999).

Sadeghi et al. (2006) apply iterative decoders to lattices constructed from codes using Construction D'.

Nested binary codes $C(H_a) \subseteq C(H_{a-1}) \subseteq \ldots \subseteq C(H_1)$ yield a lattice

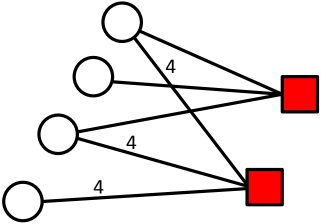
$$\Lambda_{D'} = \{\mathbf{v} \in \mathbb{Z}^n \mid H \cdot \mathbf{v}^T \equiv 0 \mod (2^{a+1}) \text{ for some } H\}.$$



Sadeghi et al. (2006) apply iterative decoders to lattices constructed from codes using Construction D'.

Nested binary codes $C(H_a) \subseteq C(H_{a-1}) \subseteq \ldots \subseteq C(H_1)$ yield a lattice

$$\Lambda_{D'} = \{\mathbf{v} \in \mathbb{Z}^n \mid H \cdot \mathbf{v}^T \equiv 0 \mod (2^{a+1}) \text{ for some } H\}.$$

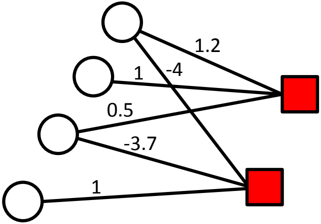


A check node is satisfied if the weighted sum of its neighbors is an integer divisible by 2^{a+1}.

Sommer et al. (2008) apply iterative decoders to lattices using the dual basis.

A lattice can be defined by its dual basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$; that is,

$$\Lambda = \{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{b}_i \text{ is an integer vector for all } i\}.$$



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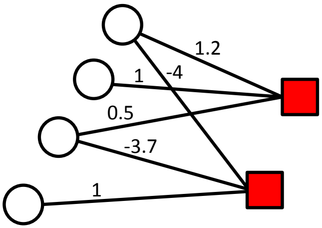

A check node is satisfied if the weighted sum of its neighbors is an integer. In this case, the message is a probability density function.

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